

Conformal Oscillator Representations of Orthogonal Lie Algebras ¹

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Abstract

The conformal transformations with respect to the metric defining the orthogonal Lie algebra $o(n, \mathbb{C})$ give rise to a one-parameter (c) family of inhomogeneous first-order differential operator representations of the orthogonal Lie algebra $o(n+2, \mathbb{C})$. Letting these operators act on the space of exponential-polynomial functions that depend on a parametric vector $\vec{a} \in \mathbb{C}^n$, we prove that the space forms an irreducible $o(n+2, \mathbb{C})$ -module for any $c \in \mathbb{C}$ if \vec{a} is not on a certain hypersurface. By partially swapping differential operators and multiplication operators, we obtain more general differential operator representations of $o(n+2, \mathbb{C})$ on the polynomial algebra \mathcal{C} in n variables. Moreover, we prove that \mathcal{C} forms an infinite-dimensional irreducible weight $o(n+2, \mathbb{C})$ -module with finite-dimensional weight subspaces if $c \notin \mathbb{Z}/2$.

Keywords: orthogonal Lie algebra; differential operator; oscillator representation; irreducible module; polynomial algebra; exponential-polynomial function.

1 Introduction

A module of a finite-dimensional simple Lie algebra is called a *weight module* if it is a direct sum of its weight subspaces. A module of a finite-dimensional simple Lie algebra is called *cuspidal* if it is not induced from its proper parabolic subalgebras. Infinite-dimensional irreducible weight modules of finite-dimensional simple Lie algebras with finite-dimensional weight subspaces have been intensively studied by the authors in [BBL], [BFL], [BHL], [BL1], [BL2], [Fs], [Fv], [M]. In particular, Fernando [Fs] proved that such modules must be cuspidal or parabolically induced. Moreover, such cuspidal modules exist only for special linear Lie algebras and symplectic Lie algebras. A similar result was independently obtained by Futorny [Fv]. Mathieu [M] proved that these cuspidal such modules are irreducible components in the tensor modules of their multiplicity-free modules with finite-dimensional modules. Although the structures of irreducible weight modules of finite-dimensional simple Lie algebra with finite-dimensional weight subspaces were essentially determined by Fernando's result in [Fs] and Mathieu's result in [M], explicit structures of such modules are not that known. It is important to find explicit natural realizations of them.

The n -dimensional conformal group with respect to Euclidean metric (\cdot, \cdot) is generated by the translations, rotations, dilations and special conformal transformations

$$\vec{x} \mapsto \frac{\vec{x} - (\vec{x}, \vec{x})\vec{b}}{(\vec{b}, \vec{b})(\vec{x}, \vec{x}) - 2(\vec{b}, \vec{x}) + 1}. \quad (1.1)$$

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Conformal groups play important roles in geometry, partial differential equations and quantum physics. The conformal transformations with respect to the metric defining $o(n, \mathbb{C})$ give rise to an inhomogeneous representation of the Lie algebra $o(n+2, \mathbb{C})$ on the polynomial algebra in n variables. Using Shen's mixed product for Witt algebras in [S] and the above representation, Zhao and the author [XZ] constructed a new functor from $o(n, \mathbb{C})$ -**Mod** to $o(n+2, \mathbb{C})$ -**Mod** and derived a condition the functor to map a finite-dimensional irreducible $o(n, \mathbb{C})$ -module to an infinite-dimensional irreducible $o(n+2, \mathbb{C})$ -module. Our general frame also gave a direct polynomial extension from irreducible $o(n, \mathbb{C})$ -modules to irreducible $o(n+2, \mathbb{C})$ -modules.

The work [XZ] lead to a one-parameter (c) family of inhomogeneous first-order differential operator (oscillator) representations of $o(n+2, \mathbb{C})$. Letting these operators act on the space of exponential-polynomial functions that depend on a parametric vector $\vec{a} \in \mathbb{C}^n$, we prove in this paper that the space forms an irreducible $o(n+2, \mathbb{C})$ -module for any $c \in \mathbb{C}$ if \vec{a} is not on a certain hypersurface. By partially swapping differential operators and multiplication operators, we obtain more general differential operator (oscillator) representations of $o(n+2, \mathbb{C})$ on the polynomial algebra \mathcal{C} in n variables. Moreover, we prove that \mathcal{C} forms an infinite-dimensional irreducible weight $o(n+2, \mathbb{C})$ -module with finite-dimensional weight subspaces if $c \notin \mathbb{Z}/2$. Our results are extensions of Howe's oscillator construction of infinite-dimensional multiplicity-free irreducible representations for $sl(n, \mathbb{C})$ (cf. [H]).

For any two integers $p \leq q$, we denote $\overline{p, q} = \{p, p+1, \dots, q\}$. Let $E_{r,s}$ be the square matrix with 1 as its (r, s) -entry and 0 as the others. Fix a positive integer n . Denote

$$A_{i,j} = E_{i,j} - E_{n+1+j, n+1+i}, \quad B_{i,j} = E_{i, n+1+j} - E_{j, n+1+i}, \quad C_{i,j} = E_{n+1+i, j} - E_{n+1+j, i} \quad (1.2)$$

for $i, j \in \overline{1, n+1}$. Then the split even orthogonal Lie algebra

$$o(2n+2, \mathbb{C}) = \sum_{i,j=1}^{n+1} (\mathbb{C}A_{i,j} + \mathbb{C}B_{i,j} + \mathbb{C}C_{i,j}). \quad (1.3)$$

Set

$$D = \sum_{r=1}^n x_r \partial_{x_r} + \sum_{s=1}^n y_s \partial_{y_s}, \quad \eta = \sum_{i=1}^n x_i y_i. \quad (1.4)$$

According to Zhao and the author's work [XZ], we have the following one-parameter generalization π_c of the conformal representation of $o(2n+2, \mathbb{C})$:

$$\pi_c(A_{i,j}) = x_i \partial_{x_j} - y_j \partial_{x_i}, \quad \pi_c(B_{i,j}) = x_i \partial_{y_j} - x_j \partial_{y_i}, \quad \pi_c(C_{i,j}) = y_i \partial_{x_j} - y_j \partial_{x_i}, \quad (1.5)$$

$$\pi_c(A_{n+1,i}) = \partial_{x_i}, \quad \pi_c(A_{n+1, n+1}) = -D - c, \quad \pi_c(B_{i, n+1}) = -\partial_{y_i}, \quad (1.6)$$

$$\pi_c(A_{i, n+1}) = \eta \partial_{y_i} - x_i(D + c), \quad \pi_c(C_{n+1, i}) = y_i(D + c) - \eta \partial_{x_i} \quad (1.7)$$

for $i, j \in \overline{1, n}$. For $\vec{a} = (a_1, a_2, \dots, a_n)^t$, $\vec{b} = (b_1, b_2, \dots, b_n)^t \in \mathbb{C}^n$, we put

$$\vec{a} \cdot \vec{x} = \sum_{i=1}^n a_i x_i, \quad \vec{b} \cdot \vec{y} = \sum_{i=1}^n b_i y_i. \quad (1.8)$$

Let $\mathcal{A} = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$ be the algebra of polynomials in $x_1, \dots, x_n, y_1, \dots, y_n$. Moreover, we set

$$\mathcal{A}_{\vec{a}, \vec{b}} = \{f e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \mid f \in \mathcal{A}\}. \quad (1.9)$$

Denote by $\pi_{c,\vec{a},\vec{b}}$ the representation π_c of $o(2n+2, \mathbb{C})$ on $\mathcal{A}_{\vec{a},\vec{b}}$.

Fix $n_1, n_2 \in \overline{1, n}$ with $n_1 \leq n_2$. Changing operators $\partial_{x_r} \mapsto -x_r$, $x_r \mapsto \partial_{x_r}$ for $r \in \overline{1, n_1}$ and $\partial_{y_s} \mapsto -y_s$, $y_s \mapsto \partial_{y_s}$ for $s \in \overline{n_2+1, n}$ in the representation π_c of $o(2n+2, \mathbb{C})$, we get another differential-operator representation $\pi_c^{n_1, n_2}$ of $o(2n+2, \mathbb{C})$ on \mathcal{A} . We call π_c and $\pi_c^{n_1, n_2}$ the *conformal oscillator representations* of $o(2n+2, \mathbb{C})$ in terms of physics terminology. In this paper, we prove:

Theorem 1. *The representation $\pi_{c,\vec{a},\vec{b}}$ of $o(2n+2, \mathbb{C})$ is irreducible for any $c \in \mathbb{C}$ if $\sum_{i=1}^n a_i b_i \neq 0$. Moreover, the representation $\pi_c^{n_1, n_2}$ of $o(2n+2, \mathbb{C})$ is irreducible for any $c \in \mathbb{C} \setminus (\mathbb{Z}/2)$, and its underlying module \mathcal{A} is an infinite-dimensional irreducible weight $o(2n+2, \mathbb{C})$ -module with finite-dimensional weight subspaces.*

Set

$$K_i = E_{0,i} - E_{n+i+1,0}, \quad K_{n+1+i} = E_{0,n+1+i} - E_{i,0} \quad \text{for } i \in \overline{1, n+1}. \quad (1.10)$$

Then the split odd orthogonal Lie algebra

$$o(2n+3, \mathbb{C}) = o(2n+2, \mathbb{C}) + \sum_{i=1}^{2n+2} \mathbb{C}K_i. \quad (1.11)$$

Moreover, we redefine

$$D = \sum_{r=0}^n x_r \partial_{x_r} + \sum_{r=1}^n y_r \partial_{y_r} \quad \eta = \frac{1}{2} x_0^2 + \sum_{i=1}^n x_i y_i. \quad (1.12)$$

According to Zhao and the author's work [XZ], we have the following one-parameter generalization of the conformal representation π_c of $o(2n+3, \mathbb{C})$: $\pi_c|_{o(2n+2, \mathbb{C})}$ is given in (1.5)-(1.7) with D and η in (1.12),

$$\pi_c(K_i) = x_0 \partial_{x_i} - y_i \partial_{x_0}, \quad \pi(K_{n+1+i}) = x_0 \partial_{y_i} - x_i \partial_{x_0} \quad \text{for } i \in \overline{1, n}, \quad (1.13)$$

$$\pi_c(K_{n+1}) = x_0(D+c) - \eta \partial_{x_0}, \quad \pi_c(K_{2n+2}) = -\partial_{x_0}. \quad (1.14)$$

Fix $n_1, n_2 \in \overline{1, n}$ with $n_1 \leq n_2$. Changing operators $\partial_{x_r} \mapsto -x_r$, $x_r \mapsto \partial_{x_r}$ for $r \in \overline{1, n_1}$ and $\partial_{y_s} \mapsto -y_s$, $y_s \mapsto \partial_{y_s}$ for $s \in \overline{n_2+1, n}$ in the above representation of $o(2n+3, \mathbb{C})$, we get another differential-operator representation $\pi_c^{n_1, n_2}$ of $o(2n+3, \mathbb{C})$. Again call the representations π_c and $\pi_c^{n_1, n_2}$ of $o(2n+3, \mathbb{C})$ *conformal oscillator representations* in terms of physics terminology.

Let $\mathcal{B} = \mathbb{C}[x_0, x_1, \dots, x_n, y_1, \dots, y_n]$ be the algebra of polynomials in $x_0, x_1, \dots, x_n, y_1, \dots, y_n$. Redenote

$$\vec{a} \cdot \vec{x} = \sum_{i=0}^n a_i x_i \quad \text{for } \vec{a} = (a_0, a_1, \dots, a_n)^t \in \mathbb{C}^{1+n}. \quad (1.15)$$

Fix $\vec{a} \in \mathbb{C}^{1+n}$, $\vec{b} \in \mathbb{C}^n$ and $n_1, n_2 \in \overline{1, n}$ with $n_1 \leq n_2$. We set

$$\mathcal{B}_{\vec{a},\vec{b}} = \{f e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \mid f \in \mathcal{B}\} \quad (1.16)$$

(cf. (1.8)). Denote by $\pi_{c,\vec{a},\vec{b}}$ the representation π_c of $o(2n+3, \mathbb{C})$ on $\mathcal{B}_{\vec{a},\vec{b}}$.

In [XZ], Zhao and the author proved that the representation $\pi_{c,\vec{0},\vec{0}}$ of $o(2n+3, \mathbb{C})$ is irreducible if and only if $c \notin -\mathbb{N}$. The following is our second main theorem in this paper.

Theorem 2. *The representation $\pi_{c, \vec{a}, \vec{b}}$ of $o(2n+3, \mathbb{C})$ is irreducible for any $c \in \mathbb{C}$ if $a_0^2 + 2 \sum_{i=1}^n a_i b_i \neq 0$. Moreover, the representation $\pi_c^{n_1, n_2}$ of $o(2n+3, \mathbb{C})$ is irreducible for any $c \in \mathbb{C} \setminus (\mathbb{Z}/2)$, and its underlying module \mathcal{B} is an infinite-dimensional irreducible weight $o(2n+3, \mathbb{C})$ -module with finite-dimensional weight subspaces.*

In Section 2, we prove Theorem 1. The proof of Theorem 2 is given in Section 3.

2 Proof of Theorem 1

First we want to prove:

Theorem 2.1. *The representation $\pi_{c, \vec{a}, \vec{b}}$ of $o(2n+2, \mathbb{C})$ is irreducible if $\sum_{i=1}^n a_i b_i \neq 0$ for any $c \in \mathbb{C}$.*

Proof. By symmetry, we may assume $a_1 \neq 0$. Let \mathcal{M} be a nonzero $o(2n+2, \mathbb{C})$ -submodule of $\mathcal{A}_{\vec{a}, \vec{b}}$. Take any $0 \neq f e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \in \mathcal{M}$ with $f \in \mathcal{A}$. Let \mathcal{A}_k be the subspace of homogeneous polynomials with degree k . Set

$$\mathcal{A}_{\vec{a}, \vec{b}, k} = \mathcal{A}_k e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \quad \text{for } k \in \mathbb{N}. \quad (2.1)$$

According to (1.6),

$$(A_{n+1, i} - a_i)(f e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}) = \partial_{x_i}(f) e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}, \quad -(B_{i, n+1} + b_i)(f e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}) = \partial_{y_i}(f) e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \quad (2.2)$$

for $i \in \overline{1, n}$. Repeatedly applying (2.2), we obtain $e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \in \mathcal{M}$. Equivalently, $\mathcal{A}_{\vec{a}, \vec{b}, 0} \subset \mathcal{M}$.

Suppose $\mathcal{A}_{\vec{a}, \vec{b}, \ell} \subset \mathcal{M}$ for some $\ell \in \mathbb{N}$. Take any $g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \in \mathcal{A}_{\vec{a}, \vec{b}, \ell}$. Since

$$(x_i \partial_{x_1} - y_1 \partial_{y_i})(g) e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}, (y_i \partial_{x_1} - y_1 \partial_{x_i})(g) e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \in \mathcal{A}_{\vec{a}, \vec{b}, \ell} \subset \mathcal{M}, \quad (2.3)$$

we have

$$A_{i, 1}(g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}) \equiv (a_1 x_i - b_i y_1) g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \equiv 0 \pmod{\mathcal{M}} \quad (2.4)$$

and

$$C_{i, 1}(g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}) \equiv (a_1 y_i - a_i y_1) g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \equiv 0 \pmod{\mathcal{M}} \quad (2.5)$$

for $i \in \overline{1, n}$ by (1.5). On the other hand, (1.4) implies

$$(D + c)(g) e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \in \mathcal{A}_{\vec{a}, \vec{b}, \ell} \subset \mathcal{M}, \quad (2.6)$$

and so (1.6) gives

$$-A_{n+1, n+1}(g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}) \equiv \left[\sum_{i=1}^n (a_i x_i + b_i y_i) \right] g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \equiv 0 \pmod{\mathcal{M}} \quad (2.7)$$

Substituting (2.4) and (2.5) into (2.7), we get

$$\left(\sum_{i=1}^n a_i b_i \right) y_1 g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \equiv 0 \pmod{\mathcal{M}}. \quad (2.8)$$

Equivalently, $y_1 g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \in \mathcal{M}$. Substituting it to (2.4) and (2.5), we obtain

$$x_i g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}, y_i g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \in \mathcal{M} \quad (2.9)$$

for $i \in \overline{1, n}$. Therefore, $\mathcal{A}_{\vec{a}, \vec{b}, \ell+1} \subset \mathcal{M}$. By induction, $\mathcal{A}_{\vec{a}, \vec{b}, \ell} \subset \mathcal{M}$ for any $\ell \in \mathbb{N}$. So $\mathcal{A}_{\vec{a}, \vec{b}} = \mathcal{M}$. Hence $\mathcal{A}_{\vec{a}, \vec{b}}$ is an irreducible $o(2n+2, \mathbb{C})$ -module. \square

Fix $n_1, n_2 \in \overline{1, n}$ with $n_1 \leq n_2$. To make notations more distinguishable, we write

$$D_{n_1, n_2} = - \sum_{i=1}^{n_1} x_i \partial_{x_i} + \sum_{r=n_1+1}^n x_r \partial_{x_r} + \sum_{j=1}^{n_2} y_j \partial_{y_j} - \sum_{s=n_2+1}^n y_s \partial_{y_s}, \quad (2.10)$$

$$\eta_{n_1, n_2} = \sum_{i=1}^{n_1} y_i \partial_{x_i} + \sum_{r=n_1+1}^{n_2} x_r y_r + \sum_{s=n_2+1}^n x_s \partial_{y_s} \quad (2.11)$$

and

$$\tilde{c} = c + n_2 - n_1 - n. \quad (2.12)$$

Then we have the following representation $\pi_c^{n_1, n_2}$ of the Lie algebra $o(2n+2, \mathbb{C})$ determined by

$$\pi_c^{n_1, n_2}(A_{i,j}) = E_{i,j}^x - E_{j,i}^y \quad (2.13)$$

with

$$E_{i,j}^x = \begin{cases} -x_j \partial_{x_i} - \delta_{i,j} & \text{if } i, j \in \overline{1, n_1}, \\ \partial_{x_i} \partial_{x_j} & \text{if } i \in \overline{1, n_1}, j \in \overline{n_1+1, n}, \\ -x_i x_j & \text{if } i \in \overline{n_1+1, n}, j \in \overline{1, n_1}, \\ x_i \partial_{x_j} & \text{if } i, j \in \overline{n_1+1, n} \end{cases} \quad (2.14)$$

and

$$E_{i,j}^y = \begin{cases} y_i \partial_{y_j} & \text{if } i, j \in \overline{1, n_2}, \\ -y_i y_j & \text{if } i \in \overline{1, n_2}, j \in \overline{n_2+1, n}, \\ \partial_{y_i} \partial_{y_j} & \text{if } i \in \overline{n_2+1, n}, j \in \overline{1, n_2}, \\ -y_j \partial_{y_i} - \delta_{i,j} & \text{if } i, j \in \overline{n_2+1, n}, \end{cases} \quad (2.15)$$

and

$$\pi_c^{n_1, n_2}(E_{i, n_1+1+j}) = \begin{cases} \partial_{x_i} \partial_{y_j} & \text{if } i \in \overline{1, n_1}, j \in \overline{1, n_2}, \\ -y_j \partial_{x_i} & \text{if } i \in \overline{1, n_1}, j \in \overline{n_2+1, n}, \\ x_i \partial_{y_j} & \text{if } i \in \overline{n_1+1, n}, j \in \overline{1, n_2}, \\ -x_i y_j & \text{if } i \in \overline{n_1+1, n}, j \in \overline{n_2+1, n}, \end{cases} \quad (2.16)$$

$$\pi_c^{n_1, n_2}(E_{n_1+1+i, j}) = \begin{cases} -x_j y_i & \text{if } j \in \overline{1, n_1}, i \in \overline{1, n_2}, \\ -x_j \partial_{y_i} & \text{if } j \in \overline{1, n_1}, i \in \overline{n_2+1, n}, \\ y_i \partial_{x_j} & \text{if } j \in \overline{n_1+1, n}, i \in \overline{1, n_2}, \\ \partial_{x_j} \partial_{y_i} & \text{if } j \in \overline{n_1+1, n}, i \in \overline{n_2+1, n}, \end{cases} \quad (2.17)$$

$$\pi_c^{n_1, n_2}(A_{n_1+1, n_1+1}) = -D_{n_1, n_2} - \tilde{c}, \quad (2.18)$$

$$\pi_c^{n_1, n_2}(A_{n_1+1, i}) = \begin{cases} -x_i & \text{if } i \in \overline{1, n_1}, \\ \partial_{x_i} & \text{if } i \in \overline{n_1+1, n}, \end{cases} \quad (2.19)$$

$$\pi_c^{n_1, n_2}(B_{i, n_1+1}) = \begin{cases} -\partial_{y_i} & \text{if } i \in \overline{1, n_2}, \\ y_i & \text{if } i \in \overline{n_2+1, n}, \end{cases} \quad (2.20)$$

$$\pi_c^{n_1, n_2}(A_{i, n_1+1}) = \begin{cases} \eta_{n_1, n_2} \partial_{y_i} - (D_{n_1, n_2} + \tilde{c} - 1) \partial_{x_i} & \text{if } i \in \overline{1, n_1}, \\ \eta_{n_1, n_2} \partial_{y_i} - x_i (D_{n_1, n_2} + \tilde{c}) & \text{if } i \in \overline{n_1+1, n_2}, \\ -\eta_{n_1, n_2} y_i - x_i (D_{n_1, n_2} + \tilde{c}) & \text{if } i \in \overline{n_2+1, n}, \end{cases} \quad (2.21)$$

$$\pi_c^{n_1, n_2}(C_{n_1+1, i}) = \begin{cases} \eta_{n_1, n_2} x_i + y_i (D_{n_1, n_2} + \tilde{c}) & \text{if } i \in \overline{1, n_1}, \\ -\eta_{n_1, n_2} \partial_{x_i} + y_i (D_{n_1, n_2} + \tilde{c}) & \text{if } i \in \overline{n_1+1, n_2}, \\ -\eta_{n_1, n_2} \partial_{x_i} + (D_{n_1, n_2} + \tilde{c} - 1) \partial_{y_i} & \text{if } i \in \overline{n_2+1, n} \end{cases} \quad (2.22)$$

for $i, j \in \overline{1, n}$.
Set

$$\mathcal{A}_{\langle k \rangle} = \text{Span}\{x^\alpha y^\beta \mid \alpha, \beta \in \mathbb{N}^n; \sum_{r=n_1+1}^n \alpha_r - \sum_{i=1}^{n_1} \alpha_i + \sum_{i=1}^{n_2} \beta_i - \sum_{r=n_2+1}^n \beta_r = k\} \quad (2.23)$$

for $k \in \mathbb{Z}$. Then

$$\mathcal{A}_{\langle k \rangle} = \{u \in \mathcal{A} \mid D_{n_1, n_2}(u) = ku\} \quad (2.24)$$

Observe that the Lie subalgebra

$$\mathcal{K} = \sum_{i,j=1}^n (\mathbb{C}A_{i,j} + \mathbb{C}B_{i,j} + \mathbb{C}C_{i,j}) \cong o(2n, \mathbb{C}). \quad (2.25)$$

With respect to the presentation $\pi_c^{n_1, n_2}$, $\mathcal{A}_{\langle k \rangle}$ forms a \mathcal{K} -module. Write

$$\mathcal{D}_{n_1, n_2} = - \sum_{i=1}^{n_1} x_i \partial_{y_i} + \sum_{r=n_1+1}^{n_2} \partial_{x_r} \partial_{y_r} - \sum_{s=n_2+1}^n y_s \partial_{x_s}. \quad (2.26)$$

Note that as operators on \mathcal{A} ,

$$\xi \eta_{n_1, n_2} = \eta_{n_1, n_2} \xi, \quad \xi \mathcal{D}_{n_1, n_2} = \mathcal{D}_{n_1, n_2} \xi \quad \text{for } \xi \in \mathcal{K}. \quad (2.27)$$

In particular,

$$\mathcal{H}_{\langle k \rangle} = \{u \in \mathcal{A}_{\langle k \rangle} \mid \mathcal{D}_{n_1, n_2}(u) = 0\} \quad (2.28)$$

forms a \mathcal{K} -module. The following result is taken from Luo and the author's work [LX2].

Lemma 2.2. *For any $n_1 - n_2 + 1 - \delta_{n_1, n_2} \geq k \in \mathbb{Z}$, $\mathcal{H}_{\langle k \rangle}$ is an irreducible \mathcal{K} -submodule and $\mathcal{A}_{\langle k \rangle} = \bigoplus_{i=0}^{\infty} \eta_{n_1, n_2}^i(\mathcal{H}_{\langle k-2i \rangle})$ is a decomposition of irreducible \mathcal{K} -submodules.*

Now we have the second result in this section.

Theorem 2.3. *The representation $\pi_c^{n_1, n_2}$ of $o(2n+2, \mathbb{C})$ on \mathcal{A} is irreducible if $c \notin \mathbb{Z}/2$.*

Proof. Let \mathcal{M} be a nonzero $o(2n+2, \mathbb{C})$ -submodule of \mathcal{A} . By (2.18) and (2.24),

$$\mathcal{M} = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}_{\langle k \rangle} \cap \mathcal{M}. \quad (2.29)$$

Thus $\mathcal{A}_{\langle k \rangle} \cap \mathcal{M} \neq \{0\}$ for some $k \in \mathbb{Z}$. If $k > n_1 - n_2 + 1 - \delta_{n_1, n_2}$, then

$$\{0\} \neq (-x_1)^{k-(n_1-n_2+1-\delta_{n_1, n_2})} (\mathcal{A}_{\langle k \rangle} \cap \mathcal{M}) = A_{n+1, 1}^{k-(n_1-n_2+1-\delta_{n_1, n_2})} (\mathcal{A}_{\langle k \rangle} \cap \mathcal{M}) \quad (2.30)$$

by (2.19), which implies $\mathcal{A}_{\langle n_1-n_2+1-\delta_{n_1, n_2} \rangle} \cap \mathcal{M} \neq \{0\}$. Thus we can assume $k \leq n_1 - n_2 + 1 - \delta_{n_1, n_2}$. Observe that the Lie subalgebra

$$\mathcal{L} = \sum_{i,j=1}^n \mathbb{C}A_{i,j} \cong sl(n, \mathbb{C}). \quad (2.31)$$

By Lemma 2.2, $\mathcal{A}_{\langle k \rangle} = \bigoplus_{i=0}^{\infty} \eta_{n_1, n_2}^i(\mathcal{H}_{\langle k-2i \rangle})$ is a decomposition of irreducible \mathcal{K} -submodules. Moreover, $\eta_{n_1, n_2}^i(\mathcal{H}_{\langle k-2i \rangle})$ are highest-weight \mathcal{L} -modules with distinct highest weights by [LX1]. Hence

$$\eta_{n_1, n_2}^i(\mathcal{H}_{\langle k-2i \rangle}) \subset \mathcal{M} \quad \text{for some } i \in \mathbb{N}. \quad (2.32)$$

Observe that

$$x_1^{-k+2i} \in \mathcal{H}_{\langle k-2i \rangle}. \quad (2.33)$$

By (2.11) and (2.20),

$$i!(-1)^i \left(\prod_{r=1}^i (-k+i+r) \right) x_1^{-k+i} = B_{1,2n+2}^i (\eta_{n_1,n_2}^i (x_1^{-k+2i})) \in \mathcal{M}. \quad (2.34)$$

Thus

$$\mathcal{H}_{\langle k-i \rangle} \subset \mathcal{M}. \quad (2.35)$$

So we can just assume

$$\mathcal{H}_{\langle k \rangle} \subset \mathcal{M}. \quad (2.36)$$

According to (2.19),

$$x_1^{-k+s} = (-1)^s A_{n+1,1}^s (x_1^{-k}) \in \mathcal{M} \quad \text{for } s \in \mathbb{N}. \quad (2.37)$$

So Lemma 2.2 gives

$$\mathcal{H}_{\langle k-s \rangle} \subset \mathcal{M} \quad \text{for } s \in \mathbb{N}. \quad (2.38)$$

For any $r \in k - \mathbb{N}$, we suppose $\eta_{n_1,n_2}^s (x_1^{-r+s}), \eta_{n_1,n_2}^s (x_1^{-r+s+1}) \in \mathcal{M}$ for some $s \in \mathbb{N}$. Applying (2.22) to it, we get

$$C_{n+1,1} [\eta_{n_1,n_2}^s (x_1^{-r+s})] = \eta_{n_1,n_2}^{s+1} (x_1^{-r+s+1}) + (r + \tilde{c}) \eta_{n_1,n_2}^s (y_1 x_1^{-r+s}) \in \mathcal{M}. \quad (2.39)$$

By (2.11) and (2.22),

$$C_{n+1,i} [\eta_{n_1,n_2}^s (x_1^{-r+s+1})] = (r - 1 + \tilde{c}) \eta_{n_1,n_2}^s (y_i x_1^{-r+s+1}) \in \mathcal{M} \quad (2.40)$$

for $i \in \overline{n_1 + 1, n_2}$. According to (2.11) and (2.21),

$$A_{i,n+1} [\eta_{n_1,n_2}^s (y_i x_1^{-r+s+1})] = \eta_{n_1,n_2}^{s+1} (x_1^{-r+s+1}) - (r + \tilde{c}) \eta_{n_1,n_2}^s (x_i y_i x_1^{-r+s+1}) \in \mathcal{M} \quad (2.41)$$

for $i \in \overline{n_1 + 1, n_2}$. Again (2.11), (2.39) and (2.41) lead to

$$(1 + r + \tilde{c} - n_2 + n_1) \eta_{n_1,n_2}^{s+1} (x_1^{-r+s+1}) \in \mathcal{M} \Rightarrow \eta_{n_1,n_2}^{s+1} (x_1^{-r+s+1}) \in \mathcal{M}. \quad (2.42)$$

By induction,

$$\eta_{n_1,n_2}^\ell (x_1^{-r+\ell}) \in \mathcal{M} \quad \text{for } \ell \in \mathbb{N}. \quad (2.43)$$

Since $\eta_{n_1,n_2}^\ell (\mathcal{H}_{\langle r-\ell \rangle}) \ni \eta_{n_1,n_2}^\ell (x_1^{-r+\ell})$ is an irreducible \mathcal{L} -module by Lemma 2.2, we have

$$\eta_{n_1,n_2}^\ell (\mathcal{H}_{\langle r-\ell \rangle}) \subset \mathcal{M} \quad \text{for } \ell \in \mathbb{N}. \quad (2.44)$$

Taking $r = m - \ell$ with $m \in k - \mathbb{N}$, we get

$$\eta_{n_1,n_2}^\ell (\mathcal{H}_{\langle m-2\ell \rangle}) \subset \mathcal{M} \quad \text{for } \ell \in \mathbb{N}. \quad (2.45)$$

According to Lemma 2.2,

$$\mathcal{A}_{\langle m \rangle} = \bigoplus_{\ell=0}^{\infty} \eta_{n_1,n_2}^\ell (\mathcal{H}_{\langle m-2\ell \rangle}) \subset \mathcal{M} \quad \text{for } m \in k - \mathbb{N}. \quad (2.46)$$

Expression (2.21) gives

$$\pi_c^{n_1, n_2}(A_{i, n+1})y_i = \begin{cases} \eta_{n_1, n_2}(y_i \partial_{y_i} + 1) - y_i \partial_{x_i}(D_{n_1, n_2} + \tilde{c} + 1) & \text{if } i \in \overline{1, n_1}, \\ \eta_{n_1, n_2}(y_i \partial_{y_i} + 1) - x_i y_i (D_{n_1, n_2} + \tilde{c} + 1) & \text{if } i \in \overline{n_1 + 1, n_2}, \end{cases} \quad (2.47)$$

$$\pi_c^{n_1, n_2}(A_{j, n+1})\partial_{y_j} = -\eta_{n_1, n_2}y_j \partial_{y_j} - x_j \partial_{y_j}(D_{n_1, n_2} + \tilde{c} + 1) \quad \text{for } j \in \overline{n_2 + 1, n}. \quad (2.48)$$

Moreover, (2.22) yields

$$\pi_c^{n_1, n_2}(C_{n+1, r})\partial_{x_r} = \eta_{n_1, n_2}x_r \partial_{x_i} + y_r \partial_{x_r}(D_{n_1, n_2} + \tilde{c} + 1) \quad \text{for } r \in \overline{1, n_1}, \quad (2.49)$$

$$\begin{aligned} & \pi_c^{n_1, n_2}(C_{n+1, s})x_s \\ = & \begin{cases} -\eta_{n_1, n_2}(x_s \partial_{x_s} + 1) + x_s y_s (D_{n_1, n_2} + \tilde{c} + 1) & \text{if } s \in \overline{n_1 + 1, n_2}, \\ -\eta_{n_1, n_2}(x_s \partial_{x_s} + 1) + x_s \partial_{y_s}(D_{n_1, n_2} + \tilde{c} + 1) & \text{if } s \in \overline{n_2 + 1, n}. \end{cases} \end{aligned} \quad (2.50)$$

Thus

$$\begin{aligned} & \sum_{i=1}^{n_2} \pi_c^{n_1, n_2}(A_{i, n+1})y_i + \sum_{j=n_2+1}^n \pi_c^{n_1, n_2}(A_{j, n+1})\partial_{y_j} \\ & - \sum_{r=1}^{n_1} \pi_c^{n_1, n_2}(C_{n+1, r})\partial_{x_r} - \sum_{s=n_1+1}^n \pi_c^{n_1, n_2}(C_{n+1, s})x_s \\ = & \eta_{n_1, n_2}(-D_{n_1, n_2} + n_2 + n - n_1 - 2(\tilde{c} + 1)) \end{aligned} \quad (2.51)$$

as operators on \mathcal{A} . Suppose that $\mathcal{A}_{\langle \ell-s \rangle} \subset \mathcal{M}$ for some $k \leq \ell \in \mathbb{Z}$ and any $s \in \mathbb{N}$. For any $f \in \mathcal{A}_{\langle \ell-1 \rangle}$, we apply the above equation to it and get

$$(1 - \ell + n_2 + n - n_1 - 2(\tilde{c} + 1))\eta_{n_1, n_2}(f) \in \mathcal{M}. \quad (2.52)$$

Since $c \notin \mathbb{Z}/2$, we have

$$\eta_{n_1, n_2}(f) \in \mathcal{M}. \quad (2.53)$$

Now for any $g \in \mathcal{A}_{\langle \ell \rangle}$, we have $\partial_{y_1}(g) \in \mathcal{A}_{\langle \ell-1 \rangle}$. By (2.21),

$$A_{1, n+1}(g) = \eta_{n_1, n_2}(\partial_{y_1}(g)) - (\ell + \tilde{c})\partial_{x_1}(g) \in \mathcal{M}. \quad (2.54)$$

Moreover, (2.53) and (2.54) yield

$$\partial_{x_1}(g) \in \mathcal{M} \quad \text{for } g \in \mathcal{A}_{\langle \ell \rangle}. \quad (2.55)$$

Since

$$\partial_{x_1}(\mathcal{A}_{\langle \ell \rangle}) = \mathcal{A}_{\langle \ell+1 \rangle}, \quad (2.56)$$

we obtain

$$\mathcal{A}_{\langle \ell+1 \rangle} \subset \mathcal{M}. \quad (2.57)$$

By induction on ℓ , we find

$$\mathcal{A}_{\langle \ell \rangle} \subset \mathcal{M} \quad \text{for } \ell \in \mathbb{Z}, \quad (2.58)$$

or equivalently, $\mathcal{A} = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{A}_{\langle \ell \rangle} = \mathcal{M}$. Thus \mathcal{A} is an irreducible $o(2n+2, \mathbb{C})$ -module. \square

Remark 2.4. The above irreducible representation depends on the three parameters $c \in \mathbb{F}$ and $m_1, m_2 \in \overline{1, n}$. It is not highest-weight type because of the mixture of multiplication operators and differential operators in (2.16), (2.17) and (2.19)-(2.22). Since \mathcal{A} is not completely reducible as a \mathcal{L} -module by [LX1] when $n \geq 2$ and $n_1 < n$, \mathcal{A} is not a unitary $o(2n+2, \mathbb{C})$ -module. Expression (2.18) shows that \mathcal{A} is a weight $o(2n+2, \mathbb{C})$ -module with finite-dimensional weight subspaces.

Theorem 1 follows from Theorem 2.1, Theorem 2.3 and the above remark.

3 Proof of Theorem 2

In this section, we prove Theorem 2. Our first result in this section is as follows.

Theorem 3.1. *The representation $\pi_{c,\vec{a},\vec{b}}$ of $o(2n+3, \mathbb{C})$ is irreducible for any $c \in \mathbb{C}$ if $a_0^2 + 2 \sum_{i=1}^n a_i b_i \neq 0$.*

Proof. Let \mathcal{B}_k be the subspace of homogeneous polynomials with degree k . Set

$$\mathcal{B}_{\vec{a},\vec{b},k} = \mathcal{B}_k e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \quad \text{for } k \in \mathbb{N} \quad (3.1)$$

(cf. (1.15) and the second equation in (1.8)). Let \mathcal{M} be a nonzero $o(2n+3, \mathbb{C})$ -submodule of $\mathcal{B}_{\vec{a},\vec{b}}$. Take any $0 \neq f e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \in \mathcal{M}$ with $f \in \mathcal{B}$. According to (1.6),

$$(A_{n+1,i} - a_i)(f e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}) = \partial_{x_i}(f) e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}, \quad -(B_{i,n+1} + b_i)(f e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}) = \partial_{y_i}(f) e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \quad (3.2)$$

for $i \in \overline{1, n}$. Moreover, the second equation in (1.14) gives

$$-(K_{2n+2} + a_0)(f e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}) = \partial_{x_0}(f) e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}. \quad (3.3)$$

Repeatedly applying (3.2) and (3.3), we obtain $e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \in \mathcal{M}$. Equivalently, $\mathcal{B}_{\vec{a},\vec{b},0} \subset \mathcal{M}$. Suppose $\mathcal{B}_{\vec{a},\vec{b},\ell} \subset \mathcal{M}$ for some $\ell \in \mathbb{N}$. Let $g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}$ be any element in $\mathcal{A}_{\vec{a},\vec{b},\ell}$.

Case 1. $a_i \neq 0$ or $b_i \neq 0$ for some $i \in \overline{1, n}$.

By symmetry, we may assume $a_1 \neq 0$. Expression (2.3) with $\mathcal{A}_{\vec{a},\vec{b},\ell}$ replaced by $\mathcal{B}_{\vec{a},\vec{b},\ell}$ implies

$$A_{i,1}(g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}) \equiv (a_1 x_i - b_i y_1) g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \equiv 0 \pmod{\mathcal{M}} \quad (3.4)$$

and

$$C_{1+i,1}(g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}) \equiv (a_1 y_i - a_i y_1) g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \equiv 0 \pmod{\mathcal{M}} \quad (3.5)$$

for $i \in \overline{1, n}$ by (1.5). Moreover, the first equation in (1.13) gives

$$K_1(g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}) \equiv (a_1 x_0 - a_0 y_1) g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \equiv 0 \pmod{\mathcal{M}} \quad (3.6)$$

because

$$(x_0 \partial_{x_1} - y_1 \partial_{x_0})(g) e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \in \mathcal{B}_{\vec{a},\vec{b},\ell} \subset \mathcal{M}. \quad (3.7)$$

On the other hand, the second equation in (1.6) with D in (1.12) gives

$$-A_{n+1,n+1}(g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}) \equiv [a_0 x_0 + \sum_{i=1}^n (a_i x_i + b_i y_i)] g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \equiv 0 \pmod{\mathcal{M}} \quad (3.8)$$

by (2.6) with $\mathcal{A}_{\vec{a},\vec{b},\ell}$ replaced by $\mathcal{B}_{\vec{a},\vec{b},\ell}$. Substituting (3.4)-(3.6) into (3.8), we get

$$(a_0^2 + 2 \sum_{i=1}^n a_i b_i) y_1 g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \equiv 0 \pmod{\mathcal{M}}. \quad (3.9)$$

Equivalently, $y_1 g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \in \mathcal{M}$. Substituting it to (3.4)-(3.6), we obtain

$$x_0 g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}, x_i g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}, y_i g e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \in \mathcal{M} \quad (3.10)$$

for $i \in \overline{1, n}$. Therefore, $\mathcal{B}_{\vec{a},\vec{b},\ell+1} \subset \mathcal{M}$. By induction, $\mathcal{B}_{\vec{a},\vec{b},\ell} \subset \mathcal{M}$ for any $\ell \in \mathbb{N}$. So $\mathcal{B}_{\vec{a},\vec{b}} = \mathcal{M}$. Hence $\mathcal{B}_{\vec{a},\vec{b}}$ is an irreducible $o(2n+3, \mathbb{C})$ -module.

Case 2. $a_0 \neq 0$ and $a_i = b_0 = 0$ for $i \in \overline{1, n}$.

Under the above assumption,

$$K_i(ge^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}) = (x_0 \partial_{x_i} - y_i \partial_{x_0} - a_0 y_i)(g)e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \in \mathcal{M} \quad (3.11)$$

and

$$K_{n+1+i}(ge^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}) = (x_0 \partial_{y_i} - x_i \partial_{x_0} - a_0 x_i)(g)e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \in \mathcal{M} \quad (3.12)$$

for $i \in \overline{1, n}$. Note

$$(x_0 \partial_{x_i} - y_i \partial_{x_0})(g)e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}, (x_0 \partial_{y_i} - x_i \partial_{x_0})(g)e^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \in \mathcal{B}_{\vec{a}, \vec{b}, \ell} \subset \mathcal{M} \quad (3.13)$$

by the inductual assumption. Thus (3.10) and (3.11) imply

$$y_i ge^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}}, x_i ge^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \in \mathcal{M} \quad \text{for } i \in \overline{1, n}. \quad (3.14)$$

Now (3.8) yields $x_0 ge^{\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{y}} \in \mathcal{M}$. So $B_{\vec{a}, \vec{b}, \ell+1} \subset \mathcal{M}$. By induction, $\mathcal{B} = \mathcal{M}$; that is, \mathcal{B} is irreducible. \square

Fix $n_1, n_2 \in \overline{1, n}$ with $n_1 \leq n_2$. Reset

$$D_{n_1, n_2} = x_0 \partial_{x_0} - \sum_{i=1}^{n_1} x_i \partial_{x_i} + \sum_{r=n_1+1}^n x_r \partial_{x_r} + \sum_{j=1}^{n_2} y_j \partial_{y_j} - \sum_{s=n_2+1}^n y_s \partial_{y_s}, \quad (3.15)$$

$$\mathcal{D}_{n_1, n_2} = \partial_{x_0}^2 - 2 \sum_{i=1}^{n_1} x_i \partial_{y_i} + 2 \sum_{r=n_1+1}^{n_2} \partial_{x_r} \partial_{y_r} - 2 \sum_{s=n_2+1}^n y_s \partial_{x_s} \quad (3.16)$$

and

$$\eta_{n_1, n_2} = \frac{x_0^2}{2} + \sum_{i=1}^{n_1} y_i \partial_{x_i} + \sum_{r=n_1+1}^{n_2} x_r y_r + \sum_{s=n_2+1}^n x_s \partial_{y_s}. \quad (3.17)$$

Then the representation $\pi_c^{n_1, n_2}$ of $o(2n+3, \mathbb{C})$ is determined as follows: $\pi_c|_{o(2n+2, \mathbb{C})}$ is given by (2.12)-(2.22) with D_{n_1, n_2} in (3.15) and η_{n_1, n_2} in (3.17), and

$$\pi_c^{n_1, n_2}(K_i) = \begin{cases} -x_0 x_i - y_i \partial_{x_0} & \text{if } i \in \overline{1, n_1}, \\ x_0 \partial_{x_i} - y_i \partial_{x_0} & \text{if } i \in \overline{n_1+1, n_2}, \\ x_0 \partial_{x_i} - \partial_{x_0} \partial_{y_i} & \text{if } i \in \overline{n_2+1, n}, \end{cases} \quad (3.18)$$

$$\pi_c^{n_1, n_2}(K_{n+1+i}) = \begin{cases} x_0 \partial_{y_i} - \partial_{x_0} \partial_{x_i} & \text{if } i \in \overline{1, n_1}, \\ x_0 \partial_{y_i} - x_i \partial_{x_0} & \text{if } i \in \overline{n_1+1, n_2}, \\ -x_0 y_i - x_i \partial_{x_0} & \text{if } i \in \overline{n_2+1, n}, \end{cases} \quad (3.19)$$

$$\pi_c^{n_1, n_2}(K_{n+1}) = x_0(D_{n_1, n_2} + \tilde{c}) - \eta_{n_1, n_2} \partial_{x_0}, \quad \pi_c^{n_1, n_2}(K_{2n+2}) = -\partial_{x_0}. \quad (3.20)$$

Note that

$$\mathcal{G} = \mathcal{K} + \sum_{i=1}^{2n+2} \mathbb{C} K_i \quad (3.21)$$

is a Lie subalgebra isomorphic to $o(2n+1, \mathbb{C})$.

Define

$$\mathcal{B}_{\langle k \rangle} = \sum_{i=0}^{\infty} \mathcal{A}_{\langle k \rangle} x_0^i. \quad (3.22)$$

Then

$$\mathcal{B}_{\langle k \rangle} = \{u \in \mathcal{B} \mid D_{n_1, n_2}(u) = ku\} \quad \text{for } k \in \mathbb{Z} \quad (3.23)$$

and

$$\mathcal{B} = \bigoplus_{k \in \mathbb{Z}} \mathcal{B}_{\langle k \rangle}. \quad (3.24)$$

Moreover,

$$\xi D_{n_1, n_2} = D_{n_1, n_2} \xi, \quad \xi \eta_{n_1, n_2} = \eta_{n_1, n_2} \xi, \quad \xi \mathcal{D}_{n_1, n_2} = \mathcal{D}_{n_1, n_2} \xi \quad \text{for } \xi \in \mathcal{G} \quad (3.25)$$

as operators on \mathcal{B} . In particular, $\mathcal{B}_{\langle k \rangle}$ forms a \mathcal{G} -module for any $k \in \mathbb{Z}$. Furthermore,

$$\mathcal{H}_{\langle k \rangle} = \{u \in \mathcal{B}_{\langle k \rangle} \mid \mathcal{D}_{n_1, n_2}(u) = 0\} \quad (3.26)$$

forms a \mathcal{G} -module. The following result is taken from Luo and the author's work [LX2].

Lemma 3.2. *For any $k \in \mathbb{Z}$, $\mathcal{H}_{\langle k \rangle}$ is an irreducible \mathcal{G} -submodule and $\mathcal{A}_{\langle k \rangle} = \bigoplus_{i=0}^{\infty} \eta_{n_1, n_2}^i(\mathcal{H}_{\langle k-2i \rangle})$ is a decomposition of irreducible \mathcal{G} -submodules.*

Now we have the second result in this section.

Theorem 3.3. *The representation $\pi_c^{n_1, n_2}$ of $o(2n+3, \mathbb{C})$ on \mathcal{A} is irreducible if $c \notin \mathbb{Z}/2$.*

Proof. Let \mathcal{M} be a nonzero $o(2n+3, \mathbb{C})$ -submodule of \mathcal{B} . By (3.23) and (2.18) with D_{n_1, n_2} in (3.15),

$$\mathcal{M} = \bigoplus_{k \in \mathbb{Z}} \mathcal{B}_{\langle k \rangle} \cap \mathcal{M}. \quad (3.27)$$

Thus $\mathcal{B}_{\langle k \rangle} \cap \mathcal{M} \neq \{0\}$ for some $k \in \mathbb{Z}$. Take the Lie subalgebra \mathcal{L} in (2.31). By Lemma 3.2, $\mathcal{B}_{\langle k \rangle} = \bigoplus_{i=0}^{\infty} \eta_{n_1, n_2}^i(\mathcal{H}_{\langle k-2i \rangle})$ is a decomposition of irreducible \mathcal{G} -submodules. Moreover, $\eta_{n_1, n_2}^i(\mathcal{H}_{\langle k-2i \rangle})$ are highest-weight \mathcal{L} -modules with distinct highest weights by [LX1]. Hence

$$\eta_{n_1, n_2}^i(\mathcal{H}_{\langle k-2i \rangle}) \subset \mathcal{M} \quad \text{for some } i \in \mathbb{N}. \quad (3.28)$$

Lemma 3.2 and the arguments in (2.33)-(2.36) show

$$\mathcal{H}_{\langle k-s \rangle} \subset \mathcal{M} \quad \text{for } s \in \mathbb{N}. \quad (3.29)$$

Suppose $\eta_{n_1, n_2}^s(x_1^{-r+s}) \in \mathcal{M}$ for any $r \in k - \mathbb{N}$ and some $s \in \mathbb{N}$. Then,

$$K_{n+1}(\eta_{n_1, n_2}^s(x_1^{-r+s+1})) = (r-1+\tilde{c})\eta_{n_1, n_2}^s(x_0 x_1^{-r+s+1}) \in \mathcal{M} \quad (3.30)$$

by (3.17) and the first equation in (3.20), which implies $\eta_{n_1, n_2}^s(x_0 x_1^{-r+s+1}) \in \mathcal{M}$. Moreover,

$$K_{n+1}(\eta_{n_1, n_2}^s(x_0 x_1^{-r+s+1})) = -\eta_{n_1, n_2}^{s+1}(x_1^{-r+s+1}) + (r+\tilde{c})\eta_{n_1, n_2}^s(x_0^2 x_1^{-r+s+1}). \quad (3.31)$$

Now (2.39) and (2.41) with η_{n_1, n_2} in (3.17), and (3.31) lead to

$$(1/2 + r + \tilde{c} - n_2 + n_1)\eta_{n_1, n_2}^{s+1}(x_1^{-r+s+1}) \in \mathcal{M} \Rightarrow \eta_{n_1, n_2}^{s+1}(x_1^{-r+s+1}) \in \mathcal{M}. \quad (3.32)$$

By induction,

$$\eta_{n_1, n_2}^\ell(x_1^{-r+\ell}) \in \mathcal{M} \quad \text{for } \ell \in \mathbb{N}, r \in k - \mathbb{N}. \quad (3.33)$$

According to Lemma 3.2,

$$\mathcal{B}_{\langle m \rangle} = \bigoplus_{\ell=0}^{\infty} \eta_{n_1, n_2}^\ell(\mathcal{H}_{\langle m-2\ell \rangle}) \subset \mathcal{M} \quad \text{for } m \in k - \mathbb{N}. \quad (3.34)$$

Observe that

$$\pi_c^{n_1, n_2}(K_{n+1})x_0 = x_0^2(D_{n_1, n_2} + \tilde{c} + 1) - \eta_{n_1, n_2}(x_0 \partial_{x_0} + 1) \quad (3.35)$$

by (3.20). Then (3.35) and (2.47)-(2.50) with η_{n_1, n_2} in (3.17) and D_{n_1, n_2} in (3.15) yield

$$\begin{aligned} & -\pi_c^{n_1, n_2}(K_{n+1})x_0 + \sum_{i=1}^{n_2} \pi_c^{n_1, n_2}(A_{i, n+1})y_i + \sum_{j=n_2+1}^n \pi_c^{n_1, n_2}(A_{j, n+1})\partial_{y_j} \\ & - \sum_{r=1}^{n_1} \pi_c^{n_1, n_2}(C_{n+1, r})\partial_{x_r} - \sum_{s=n_1+1}^n \pi_c^{n_1, n_2}(C_{n+1, s})x_s \\ & = \eta_{n_1, n_2}(1 - D_{n_1, n+2} + n_2 + n - n_1 - 2(\tilde{c} + 1)) \end{aligned} \quad (3.36)$$

as operators on \mathcal{B} . The arguments in (2.52)-(2.58) show $\mathcal{M} = \mathcal{B}$; that is, \mathcal{B} is an irreducible $o(2n+3, \mathbb{C})$ -module. \square

Remark 3.4. The above irreducible representation depends on the three parameters $c \in \mathbb{C}$ and $m_1, m_2 \in \overline{1, n}$. It is not highest-weight type because of the mixture of multiplication operators and differential operators in (2.16), (2.17), (2.19)-(2.22), (3.18) and (3.19). Since \mathcal{B} is not completely reducible as a \mathcal{L} -module by [LX1] when $n \geq 2$ and $n_1 < n$, \mathcal{B} is not a unitary $o(2n+3, \mathbb{C})$ -module. Expression (2.18) with D_{n_1, n_2} in (3.15) shows that \mathcal{B} is a weight $o(2n+2, \mathbb{C})$ -module with finite-dimensional weight subspaces.

Theorem 2 follows from Theorem 3.1, Theorem 3.3 and the above remark.

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